Poisson homology in characteristic p

Michael Zhang, Yongyi Chen
MIT PRIMES

May 21, 2011

Poisson algebras

Let A be a commutative algebra over a field $\mathbb{F}$.

POISSON ALGEBRAS

Let A be a commutative algebra over a field $\mathbb{F}$.
A **Poisson bracket** on A is a map $\{,\}: A \times A \to A$ satisfying the following properties:

Poisson algebras

Let A be a commutative algebra over a field $\mathbb{F}$.
A **Poisson bracket** on A is a map $\{,\}: A \times A \to A$ satisfying the following properties:

- Skew-symmetry: $\{x, y\} = -\{y, x\}$

Poisson algebras

Let A be a commutative algebra over a field $\mathbb{F}$.
A **Poisson bracket** on A is a map $\{,\}: A \times A \to A$ satisfying the following properties:

- Skew-symmetry: $\{x, y\} = -\{y, x\}$
- Bilinearity: $\{z, ax + by\} = a\{z, x\} + b\{z, y\}$ for all $a, b \in \mathbb{F}$

Poisson algebras

Let A be a commutative algebra over a field $\mathbb{F}$.
A **Poisson bracket** on A is a map $\{,\}: A \times A \to A$ satisfying the following properties:

- Skew-symmetry: $\{x, y\} = -\{y, x\}$
- Bilinearity: $\{z, ax + by\} = a\{z, x\} + b\{z, y\}$ for all $a, b \in \mathbb{F}$
- Jacobi Identity: $\{x, \{y, z\}\} + \{z, \{x, y\}\} + \{y, \{z, x\}\} = 0$

Poisson algebras

Let A be a commutative algebra over a field $\mathbb{F}$.
A **Poisson bracket** on A is a map $\{,\}: A \times A \to A$ satisfying the following properties:

- Skew-symmetry: $\{x, y\} = -\{y, x\}$
- Bilinearity: $\{z, ax + by\} = a\{z, x\} + b\{z, y\}$ for all $a, b \in \mathbb{F}$
- Jacobi Identity: $\{x, \{y, z\}\} + \{z, \{x, y\}\} + \{y, \{z, x\}\} = 0$
- Leibniz Rule: $\{x, yz\} = y\{x, z\} + z\{x, y\}$

Poisson algebras

Let A be a commutative algebra over a field $\mathbb{F}$.
A **Poisson bracket** on A is a map $\{,\}: A \times A \to A$ satisfying the following properties:

- Skew-symmetry: $\{x, y\} = -\{y, x\}$
- Bilinearity: $\{z, ax + by\} = a\{z, x\} + b\{z, y\}$ for all $a, b \in \mathbb{F}$
- Jacobi Identity: $\{x, \{y, z\}\} + \{z, \{x, y\}\} + \{y, \{z, x\}\} = 0$
- Leibniz Rule: $\{x, yz\} = y\{x, z\} + z\{x, y\}$

We call $(A, \{,\})$ a **Poisson algebra**.

Poisson bracket

We define a Poisson bracket on $\mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by

Poisson bracket

We define a Poisson bracket on $\mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by

- $\{x_i, x_j\} = \{y_i, y_j\} = 0;$

Poisson bracket

We define a Poisson bracket on $\mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by

- $\{x_i, x_j\} = \{y_i, y_j\} = 0$;
- $\{y_i, x_j\} = \delta_{ij} := \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$

Poisson bracket

We define a Poisson bracket on $\mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by

- $\{x_i, x_j\} = \{y_i, y_j\} = 0$;
- $\{y_i, x_j\} = \delta_{ij} := \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$

Example

$$\begin{aligned} \{xy, y^2\} &= x\{y, y^2\} + y\{x, y^2\} \\ &= 0 + y(2y\{x, y\}) \\ &= -2y^2. \end{aligned}$$

Representations

An n-dimensional **representation** of a finite group G is a homomorphism $\rho\colon G \to GL(n)$.

Representations

An n-dimensional **representation** of a finite group G is a homomorphism $\rho\colon G \to GL(n)$.

Example

Let
$$G = \mathrm{Dic}_n := \langle a, b \mid a^{2n} = 1, b^4 = 1, b^{-1}ab = a^{-1} \rangle.$$

Representations

An n-dimensional **representation** of a finite group G is a homomorphism $\rho\colon G \to GL(n)$.

Example

Let

$$G = \mathrm{Dic}_n := \langle a, b \mid a^{2n} = 1, b^4 = 1, b^{-1}ab = a^{-1} \rangle.$$

Let ω and i be a primitive $(2n)$th root of unity and a primitive 4th root of unity, respectively, in a field $\mathbb{F}$, and let $\rho\colon G \to GL(2, \mathbb{F})$ be defined by:

Representations

An n-dimensional **representation** of a finite group G is a homomorphism $\rho\colon G \to GL(n)$.

Example

Let

$$G = \mathrm{Dic}_n := \langle a, b \mid a^{2n} = 1, b^4 = 1, b^{-1}ab = a^{-1} \rangle.$$

Let ω and i be a primitive $(2n)$th root of unity and a primitive 4th root of unity, respectively, in a field $\mathbb{F}$, and let $\rho\colon G \to GL(2, \mathbb{F})$ be defined by:

$$\rho(a) = \begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Representations

An n-dimensional **representation** of a finite group G is a homomorphism $\rho\colon G \to GL(n)$.

Example

Let

$$G = \mathrm{Dic}_n := \langle a, b \mid a^{2n} = 1, b^4 = 1, b^{-1}ab = a^{-1} \rangle.$$

Let ω and i be a primitive $(2n)$th root of unity and a primitive 4th root of unity, respectively, in a field $\mathbb{F}$, and let $\rho\colon G \to GL(2, \mathbb{F})$ be defined by:

$$\rho(a) = \begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Then ρ is a representation of G.

Invariant polynomial algebras

Let $R = \mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and let G be a group acting on R.

INVARIANT POLYNOMIAL ALGEBRAS

Let $R = \mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and let G be a group acting on R.

Definition

The **invariant algebra** of R with respect to G, denoted R^G, is the subalgebra of elements $r \in R$ such that $g \cdot r = r$ for all $g \in G$.

Invariant polynomial algebras

Let $R = \mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and let G be a group acting on R.

Definition

The **invariant algebra** of R with respect to G, denoted R^G, is the subalgebra of elements $r \in R$ such that $g \cdot r = r$ for all $g \in G$.

Example

Let S_2 act on $R = \mathbb{F}[x_1, x_2, y_1, y_2]$ by permuting indices (e.g. $(12) \cdot x_1 = x_2$). Then R^{S_2} is generated by the invariants $x_1 + x_2$, $y_1 + y_2$, x_1x_2, y_1y_2 and $x_1y_1 + x_2y_2$.

Invariant polynomial algebras

Example

Let $C_n = \langle g \mid g^n = 1 \rangle$ act on $R = \mathbb{F}[x, y]$ in the following way, where ω is a primitive nth root of unity:

$$g \cdot x = \omega x \quad \text{and} \quad g \cdot y = \omega^{-1} y.$$

Invariant polynomial algebras

Example

Let $C_n = \langle g \mid g^n = 1 \rangle$ act on $R = \mathbb{F}[x, y]$ in the following way, where ω is a primitive nth root of unity:

$$g \cdot x = \omega x \quad \text{and} \quad g \cdot y = \omega^{-1} y.$$

Then R^{C_n} is generated by x^n, y^n, and xy.

Problem statement and past results

Definition

For any Poisson algebra A, we denote by $\{A, A\}$ the linear span of all elements $\{f, g\}$ for $f, g \in A$.

Problem statement and past results

Definition

For any Poisson algebra A, we denote by $\{A, A\}$ the linear span of all elements $\{f, g\}$ for $f, g \in A$.

Definition

The Poisson homology $HP_0(A)$ of a Poisson algebra A, is

$$HP_0(A) := A/\{A, A\}.$$

Problem statement and past results

Definition

For any Poisson algebra A, we denote by $\{A, A\}$ the linear span of all elements $\{f, g\}$ for $f, g \in A$.

Definition

The Poisson homology $HP_0(A)$ of a Poisson algebra A, is

$$HP_0(A) := A/\{A, A\}.$$

- P. Etingof and T. Schedler proved using algebraic geometric methods (D-modules) that for $\mathbb{F} = \mathbb{C}$ or $\mathbb{Q}$, HP_0 is finite-dimensional in many examples, including those coming from group invariants.

Problem Statement and Past Results

Definition

For any Poisson algebra A, we denote by $\{A, A\}$ the linear span of all elements $\{f, g\}$ for $f, g \in A$.

Definition

The Poisson homology $HP_0(A)$ of a Poisson algebra A, is

$$HP_0(A) := A/\{A, A\}.$$

- P. Etingof and T. Schedler proved using algebraic geometric methods (D-modules) that for $\mathbb{F} = \mathbb{C}$ or $\mathbb{Q}$, HP_0 is finite-dimensional in many examples, including those coming from group invariants.
- We compute HP_0 when $\mathbb{F} = \mathbb{F}_p$. In this case, HP_0 is infinite-dimensional.

Computations

- We form a grading

$$A/\{A,A\} := \bigoplus_{n \geq 0} A_n$$

into finite-dimensional pieces A_n consisting of homogeneous polynomials of degree n.

Computations

- We form a grading

$$A/\{A,A\} := \bigoplus_{n \geq 0} A_n$$

into finite-dimensional pieces A_n consisting of homogeneous polynomials of degree n.

Definition

We consider the **Hilbert Series** $h(HP_0;t) := \sum \dim A_n t^n$

Computations

- We form a grading

$$A/\{A,A\} := \bigoplus_{n\geq 0} A_n$$

into finite-dimensional pieces A_n consisting of homogeneous polynomials of degree n.

Definition

We consider the **Hilbert Series** $h(HP_0;t) := \sum \dim A_n t^n$

- This is just a generating function with formal variable t formed from the grading.

RESULTS FOR $\mathbb{F}[x, y]^G$

We have examined the 2-dimensional case $\mathbb{F}[x, y]^G$.

RESULTS FOR $\mathbb{F}[x, y]^G$

We have examined the 2-dimensional case $\mathbb{F}[x, y]^G$. We proved:

Results for $\mathbb{F}[x,y]^G$

We have examined the 2-dimensional case $\mathbb{F}[x,y]^G$. We proved:

Theorem

If $G = Cyc_n$ *acts by* $\begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix}$ *where* ω *is an nth root of unity, for* $p > n$, $h(HP_0(A);t) = \sum_{m=0}^{n-2} t^{2m} + \frac{t^{2p-2}(1+t^{np})}{(1-t^{2p})(1-t^{np})}$

Results for $\mathbb{F}[x,y]^G$

We have examined the 2-dimensional case $\mathbb{F}[x,y]^G$. We proved:

Theorem

If $G = Cyc_n$ *acts by* $\begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix}$ *where* ω *is an nth root of unity, for* $p > n$, $h(HP_0(A);t) = \sum_{m=0}^{n-2} t^{2m} + \frac{t^{2p-2}(1+t^{np})}{(1-t^{2p})(1-t^{np})}$

For small p coprime with n, we prove a similar, but more complicated formula.

RESULTS FOR SUBGROUPS OF $SL_2(\mathbb{C})$

Subgroups of $SL_2(\mathbb{C})$ have integers attached called "exponents" m_i, and a Coxeter number h.

Results for subgroups of $SL_2(\mathbb{C})$

Subgroups of $SL_2(\mathbb{C})$ have integers attached called "exponents" m_i, and a Coxeter number h.

Theorem

(Etingof-Gong-Pacchiano-Ren-Schedler) For subgroups G of $SL_2(\mathbb{C})$, and $A = \mathbb{C}[x,y]^G$, the Hilbert series of $HP_0(A)$ is:
$h(HP_0;t) = \sum t^{2(m_i-1)}$

Results for subgroups of $SL_2(\mathbb{C})$

Subgroups of $SL_2(\mathbb{C})$ have integers attached called "exponents" m_i, and a Coxeter number h.

Theorem

(Etingof-Gong-Pacchiano-Ren-Schedler) For subgroups G of $SL_2(\mathbb{C})$, and $A = \mathbb{C}[x,y]^G$, the Hilbert series of $HP_0(A)$ is:
$h(HP_0;t) = \sum t^{2(m_i-1)}$

Conjecture

For subgroups G of $SL_2(\mathbb{C})$, and $A = \mathbb{F}_p[x,y]^G$, the Hilbert series of $HP_0(A)$ is

$$h(HP_0(A);t) = \sum t^{2(m_i-1)} + t^{2(p-1)} \frac{1+t^h}{(1-t^a)(1-t^b)},$$

and a and b are degrees of the primary invariants.

FUTURE DIRECTIONS

- We will try to prove the afore-mentioned conjecture for subgroups of $SL_2(\mathbb{C})$. These are the dicylic group Dic_n and the exceptional groups E_6, E_7, E_8.

Future Directions

- We will try to prove the afore-mentioned conjecture for subgroups of $SL_2(\mathbb{C})$. These are the dicylic group Dic_n and the exceptional groups E_6, E_7, E_8.
- We intend to extend our analysis of HP_0 to polynomial algebras of higher dimension, such as $\mathbb{F}[x_1, x_2, y_1, y_2]^G$.

Future Directions, cont.

- In MAGMA, we computed the Poisson homology of cones of smooth plane curves. Based on these computations we make the following:

FUTURE DIRECTIONS, CONT.

- In MAGMA, we computed the Poisson homology of cones of smooth plane curves. Based on these computations we make the following:

Conjecture

Let A be the algebra $\mathbb{F}_p[x,y,z]/Q(x,y,z)$ of functions on the cone X of a smooth plane curve of degree d (that is, Q is nonsingular, and homogeneous of degree d). Then,

$$h(HP_0(A);t) = \frac{(1-t^{d-1})^3}{(1-t)^3} + t^{p+d-3} f(t^p) \quad \textit{where}$$

$$f(z) = (1-z)^{-2}\left(2g - (2g-1)z + \sum_{j=0}^{d-2} z^j\right)$$

where $g = \frac{(d-1)(d-2)}{2}$ is the genus of the curve.

Acknowledgements

- Thank you to the MIT PRIMES program for making this all possible.

ACKNOWLEDGEMENTS

- Thank you to the MIT PRIMES program for making this all possible.
- Thank you to Pavel Etingof and Travis Schedler for the proposal of the problem and their advice throughout the project.

ACKNOWLEDGEMENTS

- Thank you to the MIT PRIMES program for making this all possible.
- Thank you to Pavel Etingof and Travis Schedler for the proposal of the problem and their advice throughout the project.
- Thank you to our mentor, David Jordan, for being a great teacher, providing guidance and taking the significant time to help us out.