Poisson homology in characteristic p

Yongyi Chen, Michael Zhang
Siemens Competition

Bedford High School and Belmont High School
University of Notre Dame

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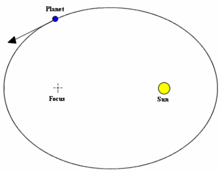
HAMILTON'S EQUATIONS

- Classical physics: systems are described in terms of **observables**, e.g. **position** x, **momentum** p.
- Can predict future if one knows the initial state of the system as a point of the **phase space**.
- Observables evolve in time via **Hamilton's equations**, which relate a system's **Hamiltonian** or **energy function** H and functions $f(\mathbf{x}, \mathbf{p})$ on a phase space.

Example: A two body problem

- Consider the earth revolving around the sun. Phase space: $\mathbb{R}^6$. Earth: $(x_1, x_2, x_3, p_1, p_2, p_3)$.



$$H = \frac{p_1^2 + p_2^2 + p_3^2}{2m} - \frac{GmM}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

where m is the mass of the Earth, M is the mass of the sun, G is the universal gravitational constant. Then

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}$$

The Poisson bracket

- If f is any observable ($f = f(\mathbf{x}, \mathbf{p})$), what is $\frac{df}{dt}$?
- From the chain rule,

$$\frac{df}{dt} = \sum_i \left(\frac{\partial f}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt} \right) = \sum_i \left(\frac{\partial f}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial x_i} \right).$$

- Can introduce the operation:

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i} \right).$$

  Then can write simply $\frac{df}{dt} = \{f, H\}$.
- We call $\{-,-\}$ a **Poisson bracket**.

THE POISSON BRACKET, CONT.

What properties does $\{-,-\}$ have?

1. Skew-symmetry: $\{x,x\} = 0 \implies \{x,y\} = -\{y,x\}$
2. Bilinearity: $\{z, ax+by\} = a\{z,x\} + b\{z,y\}$ for all $a,b \in \mathbb{F}$
3. Jacobi Identity: $\{x,\{y,z\}\} + \{z,\{x,y\}\} + \{y,\{z,x\}\} = 0$
4. Leibniz Rule: $\{x,yz\} = y\{x,z\} + z\{x,y\}$

Note that skew-symmetry implies conservation of energy:

$$\frac{dH}{dt} = \{H,H\} = 0.$$

Poisson algebras

This motivates

Definition

Let A be a commutative algebra over a field $\mathbb{F}$. A **Poisson bracket** on A is a map $\{,\}: A \times A \to A$ satisfying the following properties:

- Skew-symmetry: $\{x,x\} = 0 \implies \{x,y\} = -\{y,x\}$
- Bilinearity: $\{z, ax+by\} = a\{z,x\} + b\{z,y\}$ for all $a, b \in \mathbb{F}$
- Jacobi Identity: $\{x,\{y,z\}\} + \{z,\{x,y\}\} + \{y,\{z,x\}\} = 0$
- Leibniz Rule: $\{x, yz\} = y\{x,z\} + z\{x,y\}$

We call $(A, \{,\})$ a **Poisson algebra**.

- In such algebras, we can write down Hamilton's equations, so we can do classical mechanics in them. Poisson algebras have many applications, e.g. representation theory.

POISSON ALGEBRAS, CONT.

Example

We define a Poisson bracket on $\mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by

- $\{x_i, x_j\} = \{y_i, y_j\} = 0$;
- $\{y_i, x_j\} = \delta_{ij} := \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$

Example

Let $\mathbb{F}$ be a field and Q be a homogeneous polynomial in 3 variables over $\mathbb{F}$. We can define the Poisson bracket $\{,\}$ on $\mathbb{F}[x, y, z]/(Q)$ as follows: $\{x, y\} = \frac{\partial Q}{\partial z}$, $\{y, z\} = \frac{\partial Q}{\partial x}$, $\{z, x\} = \frac{\partial Q}{\partial y}$.

Our Project

- This project studies the internal structure of Poisson algebras. The goal is to study a basic invariant of Poisson algebras A, called the **zeroth Poisson homology**, $HP_0(A)$. This is a vector space attached to every Poisson algebra A.
- The nature of this space can be clarified by explaining the meaning of the **dual space**, $HP_0(A)^*$, i.e. the space of linear functions on $HP_0(A)$. The space $HP_0(A)^*$ is the space of linear functionals on A which are unchanged under Hamiltonian flows or vector fields for all possible Hamiltonians.

Invariant algebras

Let R be an algebra and G be a group acting on R.

Definition

The **invariant algebra** of R with respect to G, denoted R^G, is the subalgebra of elements $r \in R$ such that $g \cdot r = r$ for all $g \in G$.

Example

Let $Cyc_n = \langle g \mid g^n = 1 \rangle$ act on $R = \mathbb{F}[x, y]$ in the following way, where ω is a primitive nth root of unity:

$$g \cdot x = \omega x \quad \text{and} \quad g \cdot y = \omega^{-1} y.$$

Then R^{Cyc_n} is generated by x^n, y^n, and xy.

Problem Statement and Past Results

Definition

For any Poisson algebra A, we denote by $\{A, A\}$ the linear span of all elements $\{f, g\}$ for $f, g \in A$.

Definition

The Poisson homology $HP_0(A)$ of a Poisson algebra A, is

$$HP_0(A) := A/\{A, A\}.$$

- P. Etingof and T. Schedler proved in [3] using algebraic geometric methods (D-modules) that for $\mathbb{F} = \mathbb{C}$ or $\mathbb{Q}$, HP_0 is finite-dimensional in many examples, including those coming from group invariants.
- We compute HP_0 when $\mathbb{F} = \overline{\mathbb{F}}_p$. In this case, HP_0 is infinite-dimensional.

Computations

- We form a grading

$$A/\{A,A\} := \bigoplus_{n\geq 0} B_n$$

into finite-dimensional pieces B_n consisting of homogeneous polynomials of degree n.

Definition

We consider the **Hilbert Series** $h(HP_0;t) := \sum \dim(B_n)t^n$

- This is just a generating function with formal variable t formed from the grading.

RESULTS FOR $\mathbb{F}[x,y]^G$

We have examined the 2-dimensional case $\mathbb{F}[x,y]^G$.

Definition

- Let $\mathrm{Cyc}_n = \langle g \mid g^n = 1 \rangle$ act on $R = \mathbb{F}[x,y]$ in the following way, where ω is a primitive nth root of unity:

$$g \cdot x = \omega x \quad \text{and} \quad g \cdot y = \omega^{-1} y.$$

- Let

$$\mathrm{Dic}_n := \langle a, b \mid a^{2n} = 1, b^4 = 1, b^{-1}ab = a^{-1} \rangle.$$

Let ω and i be a primitive $(2n)$th root of unity and a primitive 4th root of unity, respectively, in a field $\mathbb{F}$. We then define the following action ρ of Dic_n on $\mathbb{F}[x,y]$:

$$\rho(a) = \begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

$\mathbb{F}[x,y]^G$, cont.

Theorem

Let $A = \overline{\mathbb{F}}_p[x,y]^G$, $G = Cyc_n$ and $p > n$. Then

$$h(HP_0(A);t) = \sum_{m=0}^{n-2} t^{2m} + \frac{t^{2p-2}(1+t^{np})}{(1-t^{2p})(1-t^{np})}$$

Theorem

Let $G = \mathrm{Dic}_n$. Suppose $p > 2n+2$. Then we have:

$$h(HP_0(A);t) = \sum_{i=0}^{n} t^{4i} + t^{2n} + t^{2p-2}\frac{1+t^{(2n+2)p}}{(1-t^{4p})(1-t^{2np})}$$

For small p coprime with n and $2n$, we can prove similar, but more complicated formulas.

SUBGROUPS OF $SL_2(\mathbb{C})$

Subgroups of $SL_2(\mathbb{C})$ have integers attached called "exponents" m_i, and a Coxeter number h.

Theorem

[2] (Etingof-Gong-Pacchiano-Ren-Schedler) For subgroups G of $SL_2(\mathbb{C})$, and $A = \mathbb{C}[x,y]^G$, the Hilbert series of $HP_0(A)$ is: $h(HP_0;t) = \sum t^{2(m_i-1)}$

Conjecture

For subgroups G of $SL_2(\mathbb{C})$, and $A = \overline{\mathbb{F}}_p[x,y]^G$, the Hilbert series of $HP_0(A)$ is

$$h(HP_0(A);t) = \sum t^{2(m_i-1)} + t^{2(p-1)} \frac{1+t^{ph}}{(1-t^{pa})(1-t^{pb})},$$

and a and b are degrees of the primary invariants.

Cones of smooth projective curves

- In MAGMA, we computed the Poisson homology of cones of smooth projective curves. Based on these computations we make the following:

Conjecture

Let A be the algebra $\overline{\mathbb{F}}_p[x,y,z]/Q(x,y,z)$ of functions on the cone X of a smooth plane curve of degree d (that is, Q is nonsingular, and homogeneous of degree d). Then,

$$h(HP_0(A);t) = \frac{(1-t^{d-1})^3}{(1-t)^3} + t^{p+d-3} f(t^p) \quad \text{where}$$

$$f(z) = (1-z)^{-2}\left(2g - (2g-1)z + \sum_{j=0}^{d-2} z^j\right)$$

where $g = \frac{(d-1)(d-2)}{2}$ *is the genus of the curve.*

CONES OF WEIGHTED PROJECTIVE CURVES

We analyze cones of weighted projective curves.

- Let $1 < A, B, C \in \mathbb{Z}_{>0}, d = LCM(A, B, C)$ and $a = d/A, b = d/B, c = d/C$.
- Consider a quasihomogeneous polynomial $Q(x, y, z)$ which has degree d where x, y, z have coprime degrees a, b, c.
- Let $h_{A,B,C,p}(t)$ be the Hilbert series of HP_0 of the surface X given by $Q(x, y, z) = 0$ over a field of characteristic p.

Conjecture

Then for large p,

$$h_{A,B,C,p}(t) = h_{A,B,C,0}(t) + t^{d-a-b-c}\phi_{A,B,C}(t^p)$$

where $\phi_{A,B,C}$ is a certain rational function independent on p, and $h_{A,B,C,0}(t)$ is the answer in characteristic 0 as computed in [4].

CONES OF WEIGHTED PROJECTIVE CURVES, CONT.

Example

Our computations focused on curves of the form $Q(x,y,z) = x^A + y^B + z^C$.

Example

- $\phi_{6,3,2} = \frac{z}{(1-z)^2}$
- $\phi_{8,4,2} = z + \frac{2z}{(1-z)(1-z^2)}$
- $\phi_{9,3,3} = z + z^2 + \frac{3z}{(1-z)(1-z^3)}$

FUTURE RESEARCH

- Prove the aforementioned conjecture for all subgroups of $SL_2(\mathbb{C})$ (the remaining groups T, C, I).
- Extend all past theorems conjectures to small p.
- Extend analysis on polynomial invariants to higher dimensions, e.g. $\mathbb{F}[x_1, x_2, y_1, y_2]$.
- Extend the results to surfaces given by $n - 2$ equations in n-dimensional spaces.

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